1.3 Curvatures

We first recall the fundamental properties of the curvature in differential geometry. Let $R = \nabla^2$ be the curvature of the Levi-Civita connection.

Recall that for $U, V, W, X \in TM$,

$$R(U, V, W, X) = g(R(U, V)X, W). \tag{1.3.1}$$

Then the curvature has the following properties:

- **Skew-symmetric**:

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Y, X, W, Z). \tag{1.3.2}$$

- **Symmetric**:

$$R(X, Y, Z, W) = R(Z, W, X, Y). \tag{1.3.3}$$

- **Bianchi's first identity**:

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0. \tag{1.3.4}$$

- **Bianchi's second identity**:

$$(\nabla_Z R)(X, Y)W + (\nabla_Y R)(Z, X)W + (\nabla_X R)(Y, Z)W = 0. \tag{1.3.5}$$

The **sectional curvature** of (V, W) is defined by

$$\sec(V, W) = \frac{R(V, W, V, W)}{g(V \wedge W, V \wedge W)}, \tag{1.3.6}$$

where

$$g(X \wedge Y, Z \wedge W) = \det \begin{pmatrix} g(X, Z) & g(X, W) \\ g(Y, Z) & g(Y, W) \end{pmatrix}. \tag{1.3.7}$$

It only depends on the plane $\pi = \operatorname{span}\{v, w\}$.

A Riemann manifold has **constant curvature** k if $\sec(\pi) = k$ for all 2-planes in T_pM.

The **Ricci curvature** of (v, w) is defined by

$$\operatorname{Ric}(V, W) = \sum_{i=1}^{n} R(e_i, V, e_i, W). \tag{1.3.8}$$

Thus Ric is a symmetric bilinear form. We adopt the language that Ric $\geq k$ if all eigenvalues of Ric are $\geq k$. That is, $Ric(V,V) \geq kg(V,V)$ for all V.

If $\mathrm{Ric}(V,W) = kg(V,W)$ for all V,W, then (M,g) is said to be an **Einstein manifold** with **Einstein constant** k. If (M,g) has constant curvature k, then (M,g) is also Einstein with Einstein constant $(n-1)k$.

The **scalar curvature** is defined by

$$\mathrm{scal} = \mathrm{tr}(\mathrm{Ric}) = 2\sum_{i<j} \sec(e_i, e_j). \tag{1.3.9}$$

Let (M,ω) be a Kähler manifold. Then the curvature is naturally extended as an endomorphism of $TM \otimes \mathbb{C}$ in a $\mathbb{C}$-linear way.

By Theorem 1.2.13, we see that $[R,J] = 0$. So for $U,V,W \in TM \otimes \mathbb{C}$,

$$R(U,V)JW = JR(U,V)W. \tag{1.3.10}$$

By (1.1.7) and (1.3.1), we have

$$R(U,V,JW,JX) = R(U,V,W,X). \tag{1.3.11}$$

So if $(W,X) \in T^{(1,0)}M \times T^{(1,0)}M$ or $T^{(0,1)}M \times T^{(0,1)}M$, $R(U,V,W,X) = 0$. Thus by (1.2.9), the curvatures are possibly non-vanishing only essentially for

$$(U,\overline{V},W,\overline{X}) \in T^{(1,0)}M \times T^{(0,1)}M \times T^{(1,0)}M \times T^{(0,1)}M. \tag{1.3.12}$$

Definition 1.3.1. Let Ric be the Ricci tensor in Riemannian geometry. For $X,Y \in TM \otimes \mathbb{C}$, we define the Ricci form $\mathrm{Ric}_\omega \in \Omega^2(M)$ by

$$\mathrm{Ric}_\omega(X,Y) = \mathrm{Ric}(JX,Y), \tag{1.3.13}$$

Definition 1.3.2. Let M be a complex manifold with triple (g,J,ω). The metric g is called **Kähler-Einstein** if (M,ω) is Kähler and Einstein. In this case, we call (M,ω) a Kähler-Einstein manifold.

Proposition 1.3.3. *If (M,ω) is a Kähler-Einstein manifold with Einstein constant k then*

$$\mathrm{Ric}_\omega = k\omega. \tag{1.3.14}$$

Proof. Our proposition follows directly from Definition 1.3.1 and (1.1.13). □

Let $e_1, \cdots, e_{2n}$ be a locally orthonormal basis of TM such that $e_{n+i} = Je_i$ for $i = 1, \cdots, n$. Let $u_i = \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}Je_i)$. Then $u_1, \cdots, u_n$ is a locally orthonormal basis of $T^{(1,0)}M$. For $\alpha \in \Omega^2(M)$, we could calculate that

$$\sqrt{-1}\sum_{i=1}^{n} \alpha(\bar{u}_i, u_i) = \sum_{i=1}^{n} \alpha(Je_i, e_i). \tag{1.3.15}$$

Proposition 1.3.4. *The Ricci form*

$$\mathrm{Ric}_\omega = \sqrt{-1}\,\mathrm{tr}^{T^{(1,0)}M}[R] = -\sqrt{-1}\partial\bar{\partial}(\log\det(h)) \in \Omega^{1,1}(M). \tag{1.3.16}$$

Proof. By Definition 1.3.1 and (1.3.15),

$$\begin{aligned}\mathrm{Ric}_\omega(X,Y) = \mathrm{Ric}(JX,Y) &= \frac{1}{2}\sum_{i=1}^{2n}(R(Je_i, JX, Je_i, Y) + R(e_i, JX, e_i, Y)) \\ = \frac{1}{2}\sum_{i=1}^{2n}(R(Y, Je_i, X, e_i) &+ R(Je_i, X, Y, e_i)) = -\frac{1}{2}\sum_{i=1}^{2n} R(X, Y, Je_i, e_i) \\ &= \sqrt{-1}\sum_{i=1}^{n} R(X, Y, \bar{u}_i, u_i) = \sqrt{-1}\sum_{i=1}^{n} h(R(X,Y)u_i, u_i) \\ &= \sqrt{-1}\,\mathrm{tr}^{T^{(1,0)}M}[R(X,Y)].\end{aligned} \tag{1.3.17}$$

From Theorem 1.2.11 and (1.2.13), on $T^{(1,0)}M$,

$$R = d\Gamma + \Gamma\wedge\Gamma = h^{-1}\bar{\partial}\partial h - h^{-1}\bar{\partial}h \wedge h^{-1}\partial h = \bar{\partial}\partial\log(h), \tag{1.3.18}$$

where h is the matrix for $h^{T^{(1,0)}M}$. Here $\log(h)$ is defined by the power series expansion

$$\log(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n} \tag{1.3.19}$$

(or the inverse of the exponential map $\exp : \mathfrak{gl}(n,\mathbb{C}) \to GL(n,\mathbb{C})$). Take care that $\log(h)$ here, which depends on the frame, is not a global function on M. But $\partial\bar{\partial}\log(h)$ is.

From (1.3.18),

$$\mathrm{Ric}_\omega = \sqrt{-1}\bar{\partial}\partial\,\mathrm{tr}^{T^{(1,0)}M}\log(h) = -\sqrt{-1}\partial\bar{\partial}\log\det(h). \tag{1.3.20}$$

The proof of our proposition is completed. ∎

Remark that in the last equality of (1.3.20), we use the matrix identity that

$$\operatorname{tr}\log(A) = \log\det(A) \tag{1.3.21}$$

holds for any complex non-degenerate matrix A.

Corollary 1.3.5. *The Ricci form* $\mathrm{Ric}_\omega \in \Omega^{1,1}(M)$ *is closed, that is*

$$d\,\mathrm{Ric}_\omega = 0. \tag{1.3.22}$$

Proof. The proposition follows from the facts that the exterior differential d is local and

$$d\partial\bar\partial = \partial^2\bar\partial + \bar\partial\partial\bar\partial = -\partial\bar\partial^2 = 0. \tag{1.3.23}$$

□

Recall that if $X, Y \in T_xM$ such that $|X| = |Y| = 1$ and $g(X, Y) = 0$, then $R(X, Y, X, Y)$ is the sectional curvature of the plane P spanned by X, Y. As in the Riemannian geometry, we want to study the Kähler manifolds with constant curvature. Unfortunately, the space form of constant positive curvature, S^{2n}, is not Kähler unless $n = 1$. So we restrict us to only study the sectional curvature of the plane which is preserved by the almost complex structure.

Definition 1.3.6. Let P be the plane in T_xM invariant by J. Let X be a unit vector in P. Then

$$K(P) = R(X, JX, X, JX) \tag{1.3.24}$$

is called the **holomorphic sectional curvature** by P.

It is easy to see that the holomorphic sectional curvature by P does not depend on the choice of X in P.

Set $U = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX)$. Then

$$K(P) = -R(U, \overline{U}, U, \overline{U}). \tag{1.3.25}$$

Definition 1.3.7. If $K(P)$ is a constant for all planes P in T_xM invariant by J and for all points $x \in M$, then M is called a **space of constant holomorphic sectional curvature**, which could be simply denoted by CHSC.

Theorem 1.3.8. *The following identities are equivalent:*

(1) a Kähler manifold M is CHSC with constant c;

(2) for any $A, B, C, D \in TM \otimes \mathbb{C}$,

$$R(A,B,C,D) = -\frac{c}{4}\big(g(A,D)g(B,C) - g(A,C)g(B,D) + g(A,JD)g(B,JC) - g(A,JC)g(B,JD) + 2g(A,JB)g(D,JC)\big); \quad (1.3.26)$$

(3) for any $U, V, W, X \in T^{(1,0)}M$,

$$R(U,\overline{V},W,\overline{X}) = -\frac{c}{2}\left(g(U,\overline{V})g(W,\overline{X}) + g(U,\overline{X})g(W,\overline{V})\right). \quad (1.3.27)$$

Proof. (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are obvious. We only need to prove (1) $\Longrightarrow$ (2).

For $A, B, C, D \in TM \otimes \mathbb{C}$, let

$$R_0(A,B,C,D) = \frac{1}{4}\big(g(A,D)g(B,C) - g(A,C)g(B,D) + g(A,JD)g(B,JC) - g(A,JC)g(B,JD) + 2g(A,JB)g(D,JC)\big) \quad (1.3.28)$$

It is easy to verify that

$$\begin{aligned} &R_0(A,B,C,D) = -R_0(B,A,C,D) = -R_0(A,B,D,C),\\ &R_0(A,B,C,D) = R_0(C,D,A,B),\\ &R_0(A,B,C,D) + R_0(B,C,A,D) + R_0(C,A,B,D) = 0,\\ &R_0(A,B,C,D) = R_0(JA,JB,C,D) = R_0(A,B,JC,JD). \end{aligned} \quad (1.3.29)$$

Recall that the curvature R also verifies (1.3.29). Since M is a CHSC with constant c,

$$R(A,JA,JA,A) = -cg(A,A)^2 = -cR_0(A,JA,JA,A). \quad (1.3.30)$$

Set $T = R - cR_0$. From (1.3.29),

$$T(A,JB,JC,D) + T(A,JD,JC,B) + T(A,JC,JD,B) \quad (1.3.31)$$

is symmetric in A, B, C, D. Since it vanishes for $A = B = C = D$ by (1.3.30), it must vanish identically.

Let $A = D$, $B = C$. We have

$$2T(A,JB,JB,A) + T(A,JA,JB,B) = 0. \quad (1.3.32)$$

From (1.3.29),

$$
\begin{aligned}
0 &= T(A, JA, JB, B) + T(JA, JB, A, B) + T(JB, A, JA, B) \\
&= T(A, JA, JB, B) - T(A, B, B, A) - T(A, JB, JB, A). \qquad (1.3.33)
\end{aligned}
$$

From (1.3.32) and (1.3.33),

$$
3T(A, JB, JB, A) + T(A, B, B, A) = 0. \qquad (1.3.34)
$$

Replacing B by JB,

$$
3T(A, B, B, A) + T(A, JB, JB, A) = 0. \qquad (1.3.35)
$$

Combining (1.3.34) and (1.3.35), we have

$$
T(A, B, B, A) = 0 \qquad (1.3.36)
$$

for any $A, B \in TM \otimes \mathbb{C}$. Thus

$$
\begin{aligned}
0 = \frac{1}{2} T(A, B+C, B+C, A) &= \frac{1}{2}(T(A, B, C, A) + T(A, C, B, A)) \\
&= T(A, B, C, A). \qquad (1.3.37)
\end{aligned}
$$

By (1.3.37),

$$
\begin{aligned}
0 = T(A+D, B, C, A+D) &= T(A, B, C, D) + T(D, B, C, A) \\
&= T(A, B, C, D) - T(C, A, B, D). \qquad (1.3.38)
\end{aligned}
$$

Replacing (A, B, C) by (C, A, B) in (1.3.38),

$$
T(C, A, B, D) = T(B, C, A, D). \qquad (1.3.39)
$$

So from (1.3.29),

$$
T(A, B, C, D) = \frac{1}{3}(T(A, B, C, D) + T(C, A, B, D) + T(B, C, A, D)) = 0 \qquad (1.3.40)
$$

for any $A, B, C, D \in TM \otimes \mathbb{C}$. That means,

$$
\begin{aligned}
R(A, B, C, D) &= -\frac{c}{4}\big(g(A, D)g(B, C) - g(A, C)g(B, D) \\
&+ g(A, JD)g(B, JC) - g(A, JC)g(B, JD) + 2g(A, JB)g(D, JC)\big). \qquad (1.3.41)
\end{aligned}
$$

The proof of our theorem is completed.

□

Corollary 1.3.9. *Let (M, ω) is a Kähler manifold, which is CHSC with constant c. Then (M, ω) is Kähler-Einstein with Einstein constant $c(n+1)/2$.*

Proof. Let $e_1, \cdots, e_{2n}$ be a locally orthonormal basis of TM such that $e_{n+i} = Je_i$ for $i = 1, \cdots, n$. By Theorem 1.3.8,

$$\begin{aligned}\mathrm{Ric}(X,Y) &= \sum_{i=1}^{n} R(e_i, X, e_i, Y) + \sum_{i=1}^{n} R(Je_i, X, Je_i, Y) \\ &= \frac{c}{4}\sum_{i=1}^{n}\big(g(X,Y) - g(X,e_i)g(Y,e_i) + 3g(X,Je_i)g(Y,Je_i)\big) \\ &+ \frac{c}{4}\sum_{i=1}^{n}\big(g(X,Y) - g(X,Je_i)g(Y,Je_i) + 3g(X,e_i)g(Y,e_i)\big) \\ &= \sum_{i=1}^{n}\frac{c}{2}\big(g(X,Y) + g(X,e_i)g(Y,e_i) + g(X,Je_i)g(Y,Je_i)\big) \\ &= \frac{(n+1)c}{2} g(X,Y). \end{aligned} \tag{1.3.42}$$

□

Corollary 1.3.10. *Let (M, g) is a Kähler manifold, which is CHSC with constant c. If $c \geq 0$ (or $c \leq 0$), the sectional curvature of (M, g) is non-negative (or non-positive).*

Proof. By Theorem 1.3.8,

$$R(A,B,A,B) = \frac{c}{4}\left(|A|^2|B|^2 - g(A,B)^2 + 3g(A,JB)^2\right). \tag{1.3.43}$$

□

Locally, set $W = w^i \frac{\partial}{\partial z_i}$ and $X = x^j \frac{\partial}{\partial z_j}$. Let $w = (w_1, \cdots, w_n)$ and $x = (x_1, \cdots, x_n)$. Then by (1.2.19), (1.3.1) and (1.3.18),

$$R(U, \overline{V}, W, \overline{X}) = -R(U, \overline{V}, \overline{X}, W) = \langle h\bar{\partial}\partial \log(h)(\overline{V}, U)w^t, \bar{x}^t\rangle. \tag{1.3.44}$$

In local coordinates, from (1.3.18) and (1.3.44),

$$\begin{aligned} R_{i\bar{j}k\bar{l}} := R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right) &= \frac{\partial^2 h_{kl}}{\partial z_i \partial \bar{z}_j} - h^{st}\frac{\partial h_{sl}}{\partial z_i}\frac{\partial h_{kt}}{\partial \bar{z}_j} \\ &= \frac{\partial^2 h_{ij}}{\partial z_k \partial \bar{z}_l} - h^{st}\frac{\partial h_{sj}}{\partial z_k}\frac{\partial h_{it}}{\partial \bar{z}_l}. \end{aligned} \tag{1.3.45}$$

Example 1.3.11 (Projective space)**.** Recall that in Example 1.1.12, we construct the Fubini-Study metric g^{FS} on $\mathbb{CP}^n$. Now we rescale the metric by

$$g_c = \frac{2}{c} g^{FS}, \quad c > 0. \tag{1.3.46}$$

Consider the unitary group $\mathrm{U}(n+1)$ on $\mathbb{C}^{n+1}$ (for any $A \in \mathrm{U}(n+1)$, $A\overline{A}^* = \mathrm{Id}$). Since $A \in \mathrm{U}(n+1)$ is linear, it induces an action on $\mathbb{CP}^n$ by

$$A([z]) = [A(z)], \quad [z] \in \mathbb{CP}^n. \tag{1.3.47}$$

By definition, $\mathrm{U}(n+1)$ action preserves the Hermitian metric on $\mathbb{C}^{n+1}$. From (1.1.33), we see that g_c is $\mathrm{U}(n+1)$-invariant. On the other hand, the $\mathrm{U}(n+1)$-action on $\mathbb{CP}^n$ is holomorphic and transversal, i.e., for any $x, y \in \mathbb{CP}^n$, there exists $A \in \mathrm{U}(n+1)$ such that $y = Ax$. So the local structure of any two points on $\mathbb{CP}^n$ is the same up to the holomorphic isometry. Thus, in order to calculate the holomorphic sectional curvature, we only need to work on one point.

At the point $\theta = 0$, we calculate from (1.1.33) that

$$g_{c,i\bar{j}} = \frac{2}{c}\delta_{ij}, \quad g_c^{i\bar{j}} = \frac{c}{2}\delta_{ij}, \quad \frac{\partial g_{c,i\bar{j}}}{\partial \theta_k} = \frac{\partial g_{c,i\bar{j}}}{\partial \bar{\theta}_k} = 0. \tag{1.3.48}$$

Moreover,

$$\begin{aligned} \frac{\partial^2 g_{c,i\bar{j}}}{\partial \theta_k \partial \bar{\theta}_l}\bigg|_{\theta=0} &= \frac{2}{c} \frac{\partial^4}{\partial \theta_i \partial \bar{\theta}_j \partial \theta_k \partial \bar{\theta}_l} \log(1 + |\theta|^2)\bigg|_{\theta=0} \\ &= \frac{2}{c} \frac{\partial}{\partial \theta_i}\bigg|_{\theta=0} \frac{(\theta_j \delta_{kl} - \delta_{jk}\theta_l)(1+|\theta|^2) - 2\theta_j(1+|\theta|^2)\delta_{kl} - \bar{\theta}_k\theta_l}{(1+|\theta|^2)^3} \\ &= -\frac{2}{c}(\delta_{ij}\delta_{kl} + \delta_{jk}\delta_{il}). \end{aligned} \tag{1.3.49}$$

By (1.3.45), we have

$$R_{i\bar{j}k\bar{l}} = -\frac{c}{2}(g_{c,i\bar{j}} g_{c,k\bar{l}} + g_{c,j\bar{k}} g_{c,i\bar{l}}). \tag{1.3.50}$$

From Theorem 1.3.8, we see that $(\mathbb{CP}^n, g_c)$ is CHSC with constant c for $c > 0$. In particular, $(\mathbb{CP}^n, g^{FS})$ is CHSC with constant 2. By Corollary 1.3.9, $\mathbb{CP}^n$ is a Kähler-Einstein manifold with Einstein constant $n+1$.

Example 1.3.12. Let $M = \mathbb{C}^n$ with trivial metric. Then the holomorphic sectional curvature vanishes.

Example 1.3.13 (Complex hyperbolic space). Let $M = B^n = \{z \in \mathbb{C}^n : |z| < 1\}$. Let

$$g_{i\bar{j}} = -\frac{\partial}{\partial z^i \partial \bar{z}^j} \log(1 - |z|^2) = \frac{(1-|z|^2)\delta_{ij} + \bar{z}_i z_j}{(1-|z|^2)^2}. \tag{1.3.51}$$

It is easy to see that the matrix $(g_{i\bar{j}})$ is positive definite. Thus it induces a metric on B^n. Then by (1.1.22),

$$\omega = -\sqrt{-1}\partial\bar{\partial}\log(1-|z|^2) = \sqrt{-1} \cdot \frac{(1-|z|^2)\delta_{ij} + \bar{z}_i z_j}{(1-|z|^2)^2} dz^i \wedge d\bar{z}^j. \tag{1.3.52}$$

is a Kähler form of B^n.

Let

$$g_c = -\frac{2}{c} g, \quad c < 0 \tag{1.3.53}$$

where g is the metric in (1.3.51). Then following the same process as in the study of projective space, we could calculate that

$$R_{i\bar{j}k\bar{l}} = \frac{c}{2}(g_{c,i\bar{j}} g_{c,k\bar{l}} + g_{c,j\bar{k}} g_{c,i\bar{l}}). \tag{1.3.54}$$

From Theorem 1.3.8, we see that $(\mathbb{CP}^n, g_c)$ is a space of constant holomorphic section curvature c for $c < 0$.

Theorem 1.3.14. *(Uniformization Theorem) For a complete Kähler manifold M of constant holomorphic sectional curvature c, its universal covering $\widetilde{M}$ is holomorphically isometric to one of the above examples.*

Proof. After rescaling, we only need to handle three cases: $c = -1, 0, 1$.

We prove $c \leq 0$ first. Let (M_c, g_c) be the Kähler manifold of constant holomorphic sectional curvature c in the above examples. Consider the exponential maps $\exp_0 : T_0 M_c \to M_c$ and $\exp_x : T_x\widetilde{M} \to \widetilde{M}$ respectively. By Corollary 1.3.10, the sectional curvatures of M_c and $\widetilde{M}$ are non-positive. By Cartan-Hadamard theorem, the exponential maps are diffeomorphisms. Here we use the complete property.

Identify both $T_0 M_c$ and $T_x\widetilde{M}$ with $\mathbb{R}^{2n}$ and define the map $\phi := \exp_x(\exp_0)^{-1}$. Since $\nabla J = 0$, we see that ϕ is holomorphic. We only need to prove that ϕ is an isometry. By Cartan-Hadamard Theorem, for any $p \in M_c$ and $X \in T_p M_c$, there exist $v, w \in \mathbb{R}^{2n}$ such that $\exp_0(v) = p$ and $d\exp_0(v)(w) = X$. If $q = \phi(p)$, $\widetilde{X} = d\phi(X)$, then $\exp_x(v) = q$ and $d\exp_x(v)(w) = \widetilde{X}$. Set

$\gamma(s,t) = \exp_0(s(v+tw))$. Let J be the corresponding Jacobi field. Then $J(1) = X$. By Jacobian equation,

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J - R(\dot\gamma, J)\dot\gamma = 0. \tag{1.3.55}$$

Take an orthonormal basis $e_1, \cdots, e_{2n}$ for T_0M_c, such that $e_1 = \dot\gamma/|\dot\gamma|$ and $e_{n+i} = Je_i$ for $i = 1, \cdots, n$. Parallel transport this basis along γ, then $\nabla_{\dot\gamma}e_i(s) = 0$ and $e_i(0) = e_i$. Write $J(s) = J^i(s)e_i(s)$, then (1.3.55) is

$$\frac{\partial^2 J^i(s)}{\partial s^2} - |\dot\gamma|^2\langle R(e_1, e_j)e_1, e_i\rangle J^j(s) = 0. \tag{1.3.56}$$

By Theorem 1.3.8,

$$\langle R(e_1, e_j)e_1, e_i\rangle = \frac{c}{4}\left(\delta_{ij} - \delta_{1i}\delta_{1j} - \delta_{i,n+1}\delta_{j,n+1} + 2\delta_{i,n+1}\delta_{j,n+1}\right). \tag{1.3.57}$$

So X is uniquely determined by v, w and c. Since $\widetilde{X}$ satisfies the same equation with the same initial values, we have $|\widetilde{X}| = |X|$. That means ϕ is an isometry.

If $c > 0$, by Corollary 1.3.10, the Ricci curvature is positive. By Myers' Theorem, we know that $\widetilde{M}$ is compact. Let $U_0 \subset \mathbb{CP}^n$ be the open subset defined in (1.1.26). Then by the same argument, we can show that ϕ is an isometry from U_0 onto its image. Since U_0 is dense in $\mathbb{CP}^n$ and $\widetilde{M}$ is compact, we can extend ϕ to all of $\mathbb{CP}^n$ so that ϕ remains an isometry.

The proof of our theorem is completed. □

Definition 1.3.15. Given two J-invariant planes P and P' in T_xM, we define the **holomorphic bisectional curvature** $H(P, P')$ by

$$H(P, P') = R(X, JX, Y, JY), \tag{1.3.58}$$

where X is a unit vector in P and Y a unit vector in P'. It is a simple matter to verify that $R(X, JX, JY, Y)$ depends only on P and P'.

Set

$$U = \frac{1}{\sqrt{2}}(X - \sqrt{-1}JX), \quad V = \frac{1}{\sqrt{2}}(Y - \sqrt{-1}JY). \tag{1.3.59}$$

Then

$$\begin{aligned} H(P, P') = R(X, JX, Y, JY) &= R(U, \overline{U}, V, \overline{V}) \\ &= R(X, Y, X, Y) + R(X, JY, X, JY). \end{aligned} \tag{1.3.60}$$

If M is CHSC with constant c, by Theorem 1.3.8 and (1.3.60),

$$H(P,P') = R(X,Y,X,Y) + R(X,JY,X,JY) = \frac{c}{2}\left(1 + g(X,Y)^2 + g(X,JY)^2\right). \quad (1.3.61)$$

It follows that, for CHSC with constant c,, the holomorphic bisectional curvatures $H(P,P')$ lie between $c/2$ and c,

$$\frac{|c|}{2} \leq |H(P,P')| \leq |c|, \quad (1.3.62)$$

where the value $c/2$ is attained when P is perpendicular to P' and the value c is attained when $P = P'$.

We state an amazing theorem related to the bisectional curvature without proof to finish this introductory chapter.

A map $f : M \to N$ between two complex manifolds is called **biholomorphic** if f is a holomorphic homeomorphism.

Theorem 1.3.16 (Siu-Yau,Mori '80)**.** *Every compact Kähler manifold of positive bisectional curvature is biholomorphic to the complex projective space.*

Remark 1.3.17. Like the sphere in Riemannian geometry, the complex projective space also has some rigidity properties. As consequences of the famous Calabi-Yau theorem, in 1977, Yau prove that

- If M is compact and Kähler, M is homeomorphic to $\mathbb{CP}^n$, then M is biholomorphic $\mathbb{CP}^n$;
- (solution of Severi conjecture)If M is a compact complex surface, M is homotopy equivalent to $\mathbb{CP}^2$, then M is biholomorphic to $\mathbb{CP}^2$.

In 1990, Libgober and Wood prove that If M is compact and Kähler, $\dim_{\mathbb{C}} M \leq 6$, M is homotopy equivalent to $\mathbb{CP}^n$, then M is biholomorphic $\mathbb{CP}^n$.

In a note of Tosatti in 2018, if there exists a compact complex manifold M diffeomorphic to S^6, then there exists a compact complex manifold M diffeomorphic to $\mathbb{CP}^3$ but not biholomorphic to it.